SOLUTION OF DIFFUSION TYPE PROBLEMS FOR EXPANDING OR CONTRACTING REGIONS WHOSE FORM VARIES WITH TIME WITHOUT PRESERVING SIMILARITY

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The method given by Grinberg in his paper [1] leads to exact solutions for a certain class of regions varying without preserving similarity. Namely, the boundary value problem is completely solved for a diffusion-type equation for the case of a parallelepiped or a cylinder (cylindrical layer) whose boundary surfaces move along the coordinate axes according to the laws $R_i(t) = \sqrt{M_i t^2 + N_i t + P_i}$ where i denotes each coordinate axis and M_i, N_i and P_i are constants depending on i.

An example of solving the problem for a cyclinder uniformly expanding or contracting with different radial and axial (vertical) velocities is given.

1. Below we show that the method presented in [1] can also be applied to regions which expand or contract without preserving similarity.

Let us consider the equation

$$\sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2} - \frac{\partial u}{\partial t} = f(x_1, x_2, x_3, t), \quad u|_{t=0} = F(x_1, x_2, x_3) \tag{1.1}$$

where x_1, x_2 and x_3 are Cartesian coordinates, f is a given function of the coordinates and time t, $u|_{t=0} = F(x_1, x_2, x_3)$ is a given initial state and the boundary of the region varies with time. We require that every function describing the law of variation of the boundary be continuous together with its first and second derivatives. Let us now introduce new variables

$$\xi_1 = x_1 / R_1, \quad \xi = x_2 / R_2, \quad \xi_2 = x_3 / R_3$$

where R_1, R_2 and R_3 describe the laws of motion of the boundaries in the x_1, x_2 and x_3 directions, respectively. In the general case we obtain, assuming that $R_1 \neq R_2 \neq R_3$, the following equation for u

$$\sum_{i=1}^{3} \frac{1}{R_i^2}\left(\frac{\partial^2 u}{\partial \xi_i^2} + \xi_i R_i R_i' \frac{\partial u}{\partial \xi_i}\right) - \frac{\partial u}{\partial t} = f(\xi_1 R_1, \xi_2 R_2, \xi_3 R_3, t)$$

$$R_i' = \frac{dR_i}{dt} \tag{1.2}$$

where the partial derivative $\partial u / \partial t$ is taken at fixed ξ_1, ξ_2 and ξ_3. Let us now replace u by a new function V

$$u = qV, \quad q = (R_1 R_2 R_3)^{-1/2} \exp\left(-\frac{1}{4} \sum_{i=1}^{3} R_i R_i' \xi_i^2\right) \tag{1.3}$$

The resulting equation for V is

$$\sum_{i=1}^{3} \frac{1}{R_i^2}\left(\frac{\partial^2 V}{\partial \xi_i^2} + {}^1/_4\, \xi_i^2 R_i'' R_i^3 V\right) - \frac{\partial V}{\partial t} = \frac{f}{q} \quad \left(R_i'' = \frac{d^2 R_i}{dt^2}\right) \tag{1.4}$$

with the initial condition

$$V|_{t=0} = \frac{F(\xi_1 R_1, \xi_2 R_2, \xi_3 R_3)}{q}\bigg|_{t=0} \tag{1.5}$$

If now

$$R_i' R_i^3 = -\alpha_i, \quad \text{or} \quad R_i = \sqrt{(A_i t + B_i)^2 - \alpha_i / A_i^2} \tag{1.6}$$

(where α_i, A_i and B_i are constants depending on i), Eq. (1.4) will assume the form

$$\sum_{i=1}^{3} \frac{1}{R_i^2}\left(\frac{\partial^2 V}{\partial \xi_i^2} - \frac{\alpha_i}{4}\xi_i^2 V\right) - \frac{\partial V}{\partial t} = \frac{f}{q} \tag{1.7}$$

This equation can always be solved by the method given in [2], provided that the boundary conditions for V are linear with constant coefficients; in the following we shall assume this to be true (*).

Thus the boundary value problem of the initial equation (1.1) can be solved completely for the case of an expanding or contracting parallelepiped without preserving similarity. The motion however must be such, that to each point $\xi_1^\circ, \xi_2^\circ, \xi_3^\circ$ of the stationary boundary surface of the problem for V there correspond points $x_i^\circ = \xi_i^\circ R_i$ $(i = 1, 2, 3)$ of the boundary surface of the initial problem for u, where $R_i(t)$ satisfies (1.6).

2. Let us consider the same problem using spherical coordinates (r, φ, z)

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial u}{\partial t} = f(r, \varphi, z, t) \tag{2.1}$$

Let $R_1(t)$ and $R_2(t)$ be the laws of motion of the boundary of the region in the radial and the z-direction. We introduce variables

$$\xi_1 = \frac{r}{R_1}, \qquad \xi_2 = \frac{z}{R_2}, \qquad u = qV \tag{2.2}$$

$$q = \frac{1}{R_1 \sqrt{R_2}} \exp\left[-\frac{1}{4}\sum_{i=1}^{2} \xi_i^2 R_i R_i'\right]$$

Equation (2.1) for V will become

$$\frac{1}{R_1^2}\left(\frac{1}{\xi_1}\frac{\partial}{\partial \xi_1}\xi_1\frac{\partial V}{\partial \xi_1} + \frac{\xi_1^2}{4} R_1^3 R_1'' V + \frac{1}{\xi_1^2}\frac{\partial^2 V}{\partial \varphi^2}\right) + \frac{1}{R_2^2}\left(\frac{\partial^2 V}{\partial \xi_2^2} + \frac{\xi_2^2}{4} R_2^3 R_2'' V\right) - $$
$$- \frac{\partial V}{\partial t} = \frac{f(\xi_1 R_1, \varphi, \xi_2 R_2, t)}{q} \tag{2.3}$$

If, as before, $R_i^3 R_i'' = -\alpha_i$, then (2.3) can be solved using the method applied to (1.7) and under the same requirements concerning the boundary conditions.

It therefore follows that the problem for a cylinder or a cylindrical layer expanding or contracting at different radial and axial (z-axis) rates, can be solved completely. The motion must however be such, that to each point $\xi_1^\circ, \varphi^\circ, \xi_2^\circ$ of the stationary boundary surface of the problem for V there correspond points $r^\circ = \xi_1^\circ R_1, \varphi^\circ, z^\circ = \xi_2 R_2$ of the boundary surface of the initial problem for u, where $R_i(t)$ satisfies (1.6).

3. Example. To solve the equation

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} - \frac{\partial u}{\partial t} = f(r, \varphi, z, t) \tag{3.1}$$

$(r \in [0, R_1(t)], \ \varphi \in [0, 2\pi], \ z \in [0, R_2(t)], \ R_i = A_i t + B_i \ (\alpha_i = 0, \ i = 1, 2))$

*) Incidentally, we note that the special condition discussed in [1] which yields the solution of the principal boundary value problem for Eq. (1.1) under the first, second or third kind conditions on a moving boundary, remains in force for the problem in question when $R_i = \sqrt{A_i t + B_i}$; in general $A_1 \neq A_2 \neq A_3$ and $B_1 \neq B_2 \neq B_3$.

with the boundary and initial conditions

$$u|_{r=0} < M,\ u|_{r=R_1(t)} = \Psi(\varphi, z, t) \qquad (M = \text{const})$$
$$\gamma_1 \frac{\partial u}{\partial z} + \gamma_2 u|_{z=0} = g(r, \varphi, t) \qquad (\gamma_1, \gamma_2 = \text{const}) \tag{3.2}$$
$$u|_{z=R_2(t)} = P(r, \varphi, t),\ u(r, \varphi, z, t) = u(r, \varphi + 2\pi, z, t),\ u|_{(t=0)} = F(r, \varphi, z)$$

we introduce new coordinates ξ_1 and ξ_2 and the function V given by (2, 2). This enables us to write the problem (3, 1) and (3, 2) in the form

$$\frac{1}{R_1^2}\left(\frac{1}{\xi_1}\frac{\partial}{\partial \xi_1}\xi_1\frac{\partial V}{\partial \xi_1} + \frac{1}{\xi_1^2}\frac{\partial^2 V}{\partial \varphi^2}\right) + \frac{1}{R_2^2}\frac{\partial^2 V}{\partial \xi_2^2} - \frac{\partial V}{\partial t} = f^*(\xi_1, \varphi, \xi_2, t) \tag{3.3}$$

with the boundary and initial conditions becoming

$$f^*(\xi_1, \varphi, \xi_2, t) = \frac{f(\xi_1 R_1, \varphi, \xi_2 R_2, t)}{q}$$
$$V|_{\xi_1=0} < M \qquad (M = \text{const}) \tag{3.4}$$
$$V|_{\xi_1=1} = \frac{\psi(\varphi, \xi_2 R_2, t)}{q|_{\xi_1=1}} \equiv \psi^*(\varphi, \xi_2, t)$$
$$\gamma_1 \frac{\partial V}{\partial \xi_2} + \gamma_2 V\Big|_{\xi_2=0} = \frac{g(\xi_1 R_1, \varphi, t)}{q|_{\xi_2=0}} \equiv g^*(\xi_1, \varphi, t)$$
$$V|_{\xi_2=1} = \frac{p(\xi_1 R_1, \varphi, t)}{q|_{\xi_2=1}} \equiv p^*(\xi_1, \varphi, t)$$
$$V(\xi_1, \varphi, \xi_2, t) = V(\xi_1, \varphi + 2\pi, \xi_2, t)$$
$$V|_{t=0} = \frac{F(\xi_1 R_1, \varphi, \xi_2 R_2)}{q|_{t=0}} \equiv F^*(\xi_1, \varphi, \xi_2)$$

We shall seek the solution of (3, 3) with conditions (3, 4) in the form of a series in terms of eigenfunctions of the corresponding homogeneous problem

$$V = \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{J_k(\nu_n^k \xi_1)}{N_{kn}^2 N_l^2}\sin\lambda_l(1-\xi_2)\,[V_{nkl}^c(t)\cos k\varphi + V_{nkl}^s(t)\sin k\varphi] \tag{3.5}$$
$$N_{kn}^2 = {}^1/_2\left[J_k'^2(\nu_n^k) + \left(1 - \frac{k^2}{(\nu_n^k)^2}\right)J_k^2(\nu_n^k)\right], \qquad N_l^2 = {}^1/_2 - \frac{1}{4\lambda_l}\sin 2\lambda_l$$
$$V_{nkl}^c = \int_0^{2\pi}\cos k\varphi\int_0^1 \xi_1 J_k(\nu_n^k \xi_1)\int_0^1 V\sin\lambda_l(1-\xi_2)\,d\xi_2 d\xi_1 d\varphi$$
$$V_{nkl}^s = \int_0^{2\pi}\sin k\varphi\int_0^1 \xi_1 J_k(\nu_n^k \xi_1)\int_0^1 V\sin\lambda_l(1-\xi_2)\,d\xi_2 d\xi_1 d\varphi$$

where $J_k(x)$ are the kth order Bessel functions of the first kind, and λ_l and ν_n^k are the roots of

$$\operatorname{tg}\lambda_l = -(\gamma_1/\gamma_2)\lambda_l, \qquad J_k(\nu_n^k) = 0$$

respectively. Equations defining V_{nkl}^c and V_{nkl}^s are obtained by multiplying (3, 3) and the initial condition in (3, 4) by weighted eigenfunctions, and integrating over the whole volume of the cylinder with the boundary conditions given in (3, 4) taken into account

$$\frac{d}{dt}V_{nkl}^c + \left(\frac{\lambda_l^2}{R_1^2} + \frac{\nu_n^{k^2}}{R_2^2}\right)V_{nkl}^c = -f_{nkl}^c - \frac{1}{R_1^2}J_k'(\nu_n^k)\,\psi_{kl}^{*c} + \frac{1}{R_2^2}\left(\frac{\sin\lambda_l}{\gamma_1}g_{kn}^{*c} - \lambda_l p_{kn}^{*c}\right)$$
$$V_{nkl}^c(0) = F_{nkl}^c \tag{3.6}$$

$$\Phi_{nkl}^{c} = \int_0^{2\pi} \cos k\varphi \int_0^1 \xi_1 J_k (\nu_n{}^k \xi_1) \int_0^1 \Phi \sin \lambda_l (1 - \xi_2)\, d\xi_2 d\xi_1 d\varphi$$

$$\psi_{kl}^{*c} = \int_0^{2\pi} \cos k\varphi \int_0^1 \psi^* \sin \lambda_l (1 - \xi_2)\, d\xi_2 d\varphi \qquad (3.7)$$

$$\Omega_{kn}^{c} = \int_0^{2\pi} \cos k\varphi \int_0^1 \Omega \xi_1 J_k (\nu_n{}^k \xi_1)\, d\xi_1 d\varphi$$

Replacing the superscript c in (3.6) and (3.7) by s and $\cos k\varphi$ and $\sin k\varphi$ in the right hand sides of (3.7), we obtain the equation for $V_{nkl}{}^s$.

Since (3.6) and the analogous equations for $V_{nkl}{}^s$ are first order linear, they can be easily integrated by quadratures. Insertion of the values of $V_{nkl}{}^c$ and $V_{nkl}{}^s$ thus obtained into (3.5), completes the solution of the problem.

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SOLUTION OF TWO-DIMENSIONAL DOUBLY-PERIODIC PROBLEMS OF THE THEORY OF STEADY VIBRATIONS OF VISCOELASTIC BODIES

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The first and second boundary value problems of the steady vibrations of a viscoelastic body occupying a domain in the form of a plane with an infinite number of identical circular holes which form an oblique-angled grating, are considered. The problems are reduced to infinite systems of algebraic equations with normal-type determinant. Reasoning of a physical nature is utilized in providing the uniqueness of the solutions of these systems.

An extensive literature has been devoted to periodic and doubly-periodic problems of plane static elasticity theory. A very detailed exposition of the results obtained is contained in the survey [1].

Let us place the origin 0_{qs} of a r_{qs}, θ_{qs} polar coordinate system at the center of each of the holes, where r_{qs} is a dimensionless coordinate expressed in fractions of the hole radius R.

Let us introduce the following notation: Γ_{qs} is the contour of the qsth hole; $R_{qs}{}^{00}$, $\theta_{qs}{}^{00}$ the polar coordinates of the pole 0_{00} in the qsth coordinate system; $U(\theta_{qs})e^{-i\omega t}$, $V(\theta_{qs})e^{-i\omega t}$ displacement components given on Γ_{qs} (the second boundary value problem); $P(\theta_{qs})e^{-i\omega t}$, $T(\theta_{qs})e^{-i\omega t}$ the normal and tangential components of the external forces